

To appear in the *Communications in Statistics - Theory and Methods*
Vol. 00, No. 00, Month 20XX, 1–6

This manuscript is to appear in
Communications in Statistics—Theory and Methods,
after a revision suggested by the referees under the new title:

Characterizations of Arcsin and Related Distributions Based on a New Generalized Unimodality
with the DOI: 10.1080/03610926.2015.1006788

Another Generalization of Unimodality

HAZHIR HOMEI*

*Department of Statistics, Faculty of Mathematical Sciences,
University of Tabriz, 29 Bahman Boulevard, P.O.Box 51666-17766, IRAN.*

(November 24, 2015)

A general characterization for α -unimodal distributions was provided by Alamatsaz (1985) who later introduced a multivariate extension of them (Alamatsaz 1993). Here, by solving the related equations, another generalization for unimodality is presented. As a result of this generalization, a simpler proof of a conjecture, as well as a characterization for generalized arcsin distributions and some generalizations of the author's earlier works, have been obtained. Last, but not the least, it is shown that some elementary methods can be more powerful than some more advanced techniques.

Keywords: Unimodality; Characterization; Generalized Arcsin Distribution.

AMS Subject Classification: 62E10; 62E15.

1. Introduction

Unimodality property is a long standing problem in distribution theory, and has been studied by several authors; see [1] and its references. The random variable Z is called unimodal when there are independent random variables X and U such that

$$Z \stackrel{d}{=} U \cdot X, \quad (1)$$

where U has a uniform distribution on $[0, 1]$. Several generalizations of (1) has been studied in the literature (see e.g. [2]); here we generalized it as follows

$$S_n \stackrel{d}{=} \langle \mathbf{R}, \mathbf{X} \rangle, \quad (2)$$

where $\mathbf{R} = (R_1, \dots, R_n)$ is the Dirichlet random vector (and so $\sum_{i=1}^n R_i = 1$) and $\mathbf{X} = (X_1, \dots, X_n)$ is an arbitrary random vector which is independent from \mathbf{R} ; the symbol $\langle \cdot, \cdot \rangle$ denotes the inner product. Our main motivation for this generalization was solving the forthcoming conjecture.

*Corresponding author. Email: homei@tabrizu.ac.ir

1.1 A Conjecture On Arcsin Distributions

Consider the inner product of two independent random vectors $\mathbf{R} = (R_1, \dots, R_n)$ and $\mathbf{X} = (X_1, \dots, X_n)$ defined by

$$S_n \stackrel{d}{=} \langle \mathbf{R}, \mathbf{X} \rangle = \sum_{i=1}^n R_i \cdot X_i \quad (n \geq 2).$$

Now, the components of the random vector \mathbf{R} can be defined as $R_i = U_{(i)} - U_{(i-1)}$ (for $i = 1, \dots, n-1$ and $R_n = 1 - \sum_{i=1}^{n-1} R_i$) where $U_{(1)}, \dots, U_{(n-1)}$ are order statistics of a random sample U_1, \dots, U_n from a uniform distribution on $[0, 1]$ with $U_{(0)} = 0$ and $U_{(n)} = 1$. Note that the distribution of $\mathbf{R} = (R_1, \dots, R_n)$ is the Dirichlet distribution $D_n(1, \dots, 1)$; see (Wang et. al. 2011).

Conjecture 1.1 If the random variables X_1, \dots, X_n are independent and have common Arcsin distribution on $(-a, a)$, then S_n will have a power semicircle distribution on $(-a, a)$ with $\lambda = \frac{(n-1)}{2}$, i.e., $f(x; \lambda, a) = \frac{1}{\sqrt{\pi a^{2\lambda}}} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\frac{1}{2})} (a^2 - x^2)^{\lambda-\frac{1}{2}} \quad (|x| < a)$ (see the conclusions of [10]). \blacklozenge

This conjecture had been proved first for the cases of $n = 2, 3, 4$ and later for all n 's; see [9]. In this short note we prove the above conjecture, and generalize it to generalized arcsin distributions, by employing simple methods of analysis based on first principles which is appropriate for classroom use in advanced undergraduate or elementary graduate courses in probability and statistics.

2. The Main Result

In order to prove our main result, we need the following lemma.

LEMMA 2.1 For all positive integers $r \in \mathbb{N}$, we have

$$\sum_{i_1 + \dots + i_n = r} \binom{r}{i_1, i_2, \dots, i_n} \frac{\Gamma(\frac{1}{2} + i_1)}{\Gamma(\frac{1}{2})} \dots \frac{\Gamma(\frac{1}{2} + i_n)}{\Gamma(\frac{1}{2})} = \frac{\Gamma(\frac{n}{2} + r)}{\Gamma(\frac{n}{2})}.$$

Proof. Let the distribution of $f(x|\mathbf{p})$ be multinomial with the parameters $\mathbf{p} = (p_1, \dots, p_n)$, and assume that $\mathbf{p} = (p_1, \dots, p_n)$ has Dirichlet distribution $D_n(\frac{1}{2}, \dots, \frac{1}{2})$. So, the distribution of $f(x)$ can be calculated, and the lemma is proved considering the fact that the sum of $f(x)$ on its support equals to one. The function $f(x)$ is called Dirichlet-Multinomial distribution (see chapters 6 and 7 of [14]). \blacksquare

THEOREM 2.2 Assume that the random variables X_1, \dots, X_n are independent and have common Arcsin distribution on $(-a, a)$. Then S_n will have a power semicircle distribution on $(-a, a)$ with $\lambda = \frac{n-1}{2}$.

Proof. Since, for any σ and ξ , we have

$$\sigma S_n + \xi = \sum_{i=1}^n R_i(\sigma X_i + \xi), \quad (3)$$

then without loss of generality we can assume that $a = 1$. We find the r^{th} moment of S_n as follows:

$$E(S_n^r) = \sum_{i_1 + \dots + i_n = r} \frac{r!}{i_1! \dots i_n!} E(R_1^{i_1} \dots R_n^{i_n}) E(X_1^{i_1}) \dots E(X_n^{i_n}).$$

By using the Dirichlet distribution, we have

$$E(S_n^r) = \sum_{i_1 + \dots + i_n = r} \frac{r!}{i_1! \dots i_n!} (n-1)! \frac{\Gamma(i_1+1) \dots \Gamma(i_n+1)}{\Gamma(r+n)} E(X_1^{i_1}) \dots E(X_n^{i_n}). \quad (4)$$

One can show that

$$E(X_j^{i_j}) = \frac{1}{2} \frac{(1 + (-1)^{i_j}) \Gamma(\frac{1}{2} + \frac{i_j}{2})}{\sqrt{\pi} \Gamma(1 + \frac{i_j}{2})}, \text{ for } j = 1, \dots, n,$$

(see page 153 of [3]). So, $E(S_n^r) =$

$$\sum_{i_1 + \dots + i_n = r} \frac{r!}{i_1! \dots i_n!} (n-1)! \frac{\Gamma(i_1+1) \dots \Gamma(i_n+1)}{\Gamma(r+n)}$$

$$\frac{1}{2} \frac{(1 + (-1)^{i_1}) \Gamma(\frac{1}{2} + \frac{i_1}{2})}{\sqrt{\pi} \Gamma(1 + \frac{i_1}{2})} \dots \frac{1}{2} \frac{(1 + (-1)^{i_n}) \Gamma(\frac{1}{2} + \frac{i_n}{2})}{\sqrt{\pi} \Gamma(1 + \frac{i_n}{2})}.$$

Since Arcsin distribution is symmetric about zero, the r^{th} moment is zero for odd r . Now we note that for even $r (= 2k)$ if $i_1 + \dots + i_n = r = 2k$ and if i_j is odd then $1 + (-1)^{i_j} = 0$, and so the corresponding summand will equal to zero. Hence, we assume all i_j 's to be even and so we write $2i_j$ in place of i_j . Thus,

$$E(S_n^{2k}) = \sum_{2i_1 + \dots + 2i_n = 2k} \frac{(2k)!}{(2i_1)! \dots (2i_n)!} (n-1)!$$

$$\frac{\Gamma(2i_1+1) \dots \Gamma(2i_n+1)}{\Gamma(2k+n)} \frac{1}{2} \frac{2\Gamma(\frac{1}{2} + i_1)}{\Gamma(\frac{1}{2}) \Gamma(1 + i_1)} \dots \frac{1}{2} \frac{2\Gamma(\frac{1}{2} + i_n)}{\Gamma(\frac{1}{2}) \Gamma(1 + i_n)}$$

$$= \frac{(2k)!(n-1)!}{\Gamma(2k+n)} \sum_{i_1 + \dots + i_n = k} \frac{1}{i_1! \dots i_n!} \frac{\Gamma(\frac{1}{2} + i_1)}{\Gamma(\frac{1}{2})} \dots \frac{\Gamma(\frac{1}{2} + i_n)}{\Gamma(\frac{1}{2})}$$

$$= \frac{(2k)!(n-1)!}{\Gamma(2k+n)k!} \sum_{i_1 + \dots + i_n = k} \frac{k!}{i_1! \dots i_n!} \frac{\Gamma(\frac{1}{2} + i_1)}{\Gamma(\frac{1}{2})} \dots \frac{\Gamma(\frac{1}{2} + i_n)}{\Gamma(\frac{1}{2})}.$$

By using Lemma 2.1 we find that

$$E(S_n^{2k}) = \frac{(2k)!(n-1)!}{\Gamma(2k+n)k!} \frac{\Gamma(\frac{n}{2} + k)}{\Gamma(\frac{n}{2})}.$$

Using the properties of gamma function we finally obtain

$$E(S_n^r) = \begin{cases} 0 & \text{if } r = 2k + 1, \\ \frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{1}{2})}{\sqrt{\pi}\Gamma(k+\frac{n}{2}+\frac{1}{2})} & \text{if } r = 2k. \end{cases}$$

It can be easily shown that this is the r^{th} moment of the power semicircle distribution (see [3]). Since S_n is a bounded random variable, its distribution is uniquely determined by its moments (Carleman's Theorem, see e.g. [5]). Thus the proof is complete. ■

Remark 2.3 Using the equation (3) and choosing suitable σ and ξ we can assume that the support of all X_i 's are $[0, 1]$ in which case the obtained moments are well-known Beta distributions. ♦

3. The Case of Common Distributions

We note that in case all X_i 's have a common distribution, Theorem 2.2 provides a characterization of Beta distributions, which is not studied before (see [9] and its references).

Remark 3.1 When X_i 's have a common distribution, their moments can be derived from the moments of S_n by using (4) noting that

$$\binom{r+n-1}{r} E(S_n^r) = \sum_{i_1+\dots+i_n=r} E(X_1^{i_1}) \cdots E(X_n^{i_n})$$

implies

$$\binom{r+n-1}{r} E(S_n^r) = E(X_1^r) + \cdots + E(X_n^r) + \sum_{i_1+\dots+i_n=r, i_1, \dots, i_n \neq r} E(X_1^{i_1}) \cdots E(X_n^{i_n})$$

and so

$$\binom{r+n-1}{r} E(S_n^r) = nE(X_i^r) + \mathcal{F}\left(E(X_i), E(X_i^2), \dots, E(X_i^{r-1})\right)$$

for a function \mathcal{F} , which finally gives us

$$E(X_i^r) = \frac{1}{n} \binom{n+r-1}{r} E(S_n^r) - \mathfrak{F}\left(E(S_n), E(S_n^2), \dots, E(S_n^{r-1})\right)$$

for a function \mathfrak{F} with $r = 1, 2, \dots$ successively. Then the distribution of X_i is characterized (since X_i 's have bounded support). ♦

Very similarly to Theorem 2.2 we can characterize the generalized Arcsin distributions as follows. Before that we need a lemma; for the definitions see e.g. [3].

LEMMA 3.2 *For all positive integers $r \in \mathbb{N}$, we have*

$$\sum_{i_1+\dots+i_n=r} \binom{r}{i_1, i_2, \dots, i_n} \frac{\Gamma(a_1+i_1)}{\Gamma(a_1)} \cdots \frac{\Gamma(a_n+i_n)}{\Gamma(a_n)} = \frac{\Gamma(r+\sum_{i=1}^n a_i)}{\Gamma(\sum_{i=1}^n a_i)}.$$

Proof. Just like the proof of Lemma 2.1 using the Dirichlet-Multinomial distribution in chapters 6 and 7 of [14]. ■

THEOREM 3.3 *Assume that the random variables X_1, \dots, X_n are independent and have common distribution on $(-a, a)$. Then S_n will have a $\text{Beta}(n\alpha, n(1 - \alpha), -a, 2a)$ distribution on $(-a, a)$ if and only if X_1, \dots, X_n have generalized $\text{Arcsin}(\alpha)$ distributions.*

Proof. By Lemma 3.2 the theorem is proved in the lines of the proof of Theorem 2.2 and Remark 3.1. ■

Remark 3.4 Though this Theorem (3.3) only applies under some very specific assumptions, it is a wild generalization of Conjecture 1.1. Also, some results of [13] and [8, 10] are special cases of Theorem 3.3. ◆

Remark 3.5 We note that having the bounded support in this work plays an essential role; but in case all X_i 's have Cauchy distributions, by using the characteristic function (i.e., the Fourier transform) and conditional expectation one can overcome this problem; see [9] and the examples of its references. ◆

Remark 3.6 In the moments method finding the desired distribution may need having sufficient information about the solution of the problem. Of course in using the method of [10] one should know the Stieltjes transform of the distribution in question, but in the moments method one can approach the solution by the guesses resulted from calculating the moments sequentially. ◆

Remark 3.7 The moments method that we used depends on the distributions which are to be characterized by their moments. This holds for all the cases studied in the references of [9], because all their distributions have bounded support. Though, some distributions which do not have bounded support could be characterized by their moments. So, this method can be applied to a large family of distributions, provided that the uniqueness conditions are satisfied. It is remarkable that still many researches in various fields (from 1990 until now) use the moments method for calculating the distributions; see for example [4, 6, 7, 11, 12] among others. ◆

4. Conclusions

The method of this article provides an elementary and direct way for calculating the distribution of the inner product of certain random vectors. So, this goes to say that the Stieltjes transform is neither the only nor the best way for calculating the distribution of the inner products of different random vectors, as long as there is no single example which cannot be handled by our proposed method. Also, this method can be extended for the cases that X_1, \dots, X_n have Beta distributions, which is intended to be studied in a future paper.

References

- [1] ALAMATSAZ, M.H. (1985). A Note on an Article by Artikis, *Acta Mathematica Hungarica* 45:1-2, 159–162.
- [2] ALAMATSAZ, M.H. (1993). On Characterizations of Exponential and Gamma Distributions, *Statistics and Probability Letters* 17:4, 315–319.

- [3] BALAKRISHNAN, N. and NEVZOROV, V.B. (2003). *A Primer on Statistical Distributions*, John Wiley and Sons.
- [4] BARRERAA, J. and HUILLET, T. (2004). On Random Splitting of the Interval, *Statistics and Probability Letters* 66:3, 237–250.
- [5] CHUNG, K.L. (2001). *A Course in Probability Theory*, Academic Press, 3rd ed.
- [6] JOHNSON, N.L. and KOTZ, S. (1990). Randomly Weighted Averages: Some Aspects and Extensions, *The American Statistician* 44:3, 245–249.
- [7] JOHNSON, N.L. and KOTZ, S. (1995). Use of Moments in Studies of Limit Distributions Arising from Iterated Random Subdivisions of an Interval, *Statistics and Probability Letters* 24:2, 111–119.
- [8] HOMEI, H. (2012). Randomly Weighted Averages with Beta Random Proportions. *Statistics and Probability Letters* 82:8, 1515–1520.
- [9] HOMEI, H. (2013). Directly Specifying the Power Semicircle Distribution, arXiv:1307.4860, available on the net at <http://arxiv.org/abs/1307.4860>
- [10] HOMEI, H. (2014). A Novel Extension of Randomly Weighted Average, *Statistical Papers*, to appear. doi: 10.1007/s00362-014-0615-5
- [11] STOYANOV, J. and LIN, G.D. (2011). Mixtures of Power Series Distributions: identifiability via uniqueness in problems of moments, *Annals of the Institute of Statistical Mathematics* 63:2, 291–303.
- [12] STOYANOV, J. and LIN, G.D. (2013). Hamburger Moment Problem for Powers and Products of Random Variables, *Journal of Statistical Planning and Inference*, to appear. doi: 10.1016/j.jspi.2013.11.002
- [13] VAN ASSCHE, W. (1987). A Random Variable Uniformly Distributed Between Two Independent Random Variables. *Sankhyā: The Indian Journal of Statistics, Series A* 49:2, 207–211.
- [14] WANG, K.N., TIAN, G.L. and TANG, M.L. (2011). *Dirichlet and Related Distributions: Theory, Methods and Applications*, John Wiley and Sons.